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Diffusion in oscillatory pipe flow

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The rate of mass transfer of a diffusing substance along a pipe is augmented by an oscillatory motion of the ambient fluid in the pipe. The increase of the flux is evaluated for the cases of a circular pipe and of a two-dimensional channel. Results are given for a general cross-section in the limiting cases of slow and fast oscillations of the flow.

1. Introduction

Taylor (1953) showed that, when a small quantity of a diffusing substance is introduced into a fluid flowing along a circular pipe, the ultimate spreading of the resultant cloud of the substance is enhanced by the flow of the fluid. This is due to the variation of the velocity of flow over the cross-section of the pipe, which causes transverse diffusion to be effective in dispersing the cloud.

A similar effect occurs when the flow is oscillatory, as pointed out by Bowden (1965). It will be shown that a solution of the concentration equation can be obtained if there is, on average, a uniform gradient of concentration along the pipe, which can have arbitrary cross-section. A special case of this (details of which are given in §4) was studied by Farrell & Larson (1973). The resultant flux of the diffusing substance depends on this cross-section, and can be evaluated analytically for any frequency of oscillation if the pipe is circular, or if it is a two-dimensional channel. It is also possible to study the general behaviour of the flux for an arbitrary cross-section in the limiting cases of slow and fast oscillations of the flow.

The results given here have already been used by Chatwin (1975). Experiments described in the companion paper (Joshi *et al.* 1983) show good agreement with the present theory.

2. General theory

It will be assumed that the diffusing substance is a passive contaminant, of which the concentration is so small that the physical properties of the fluid and the diffusivity κ of the contaminant may be taken as constant. The flow is assumed to be entirely in the z-direction, and to take place within a pipe of uniform cross-section S, which is bounded by the curve B. The pressure gradient

$$\frac{\partial p}{\partial z} = -P\cos\omega t \tag{1}$$

gives rise to a velocity distribution

$$w(x,y,t) = \mathrm{Re}\,\{f(x,y)\,\mathrm{e}^{\mathrm{i}\omega t}\}, \tag{2}$$

where

$$\mathrm{i}\omega f = \frac{P}{\rho} + \nu\nabla^2 f \quad \text{in } S, \tag{3}$$

with

$$f = 0 \quad \text{on } B. \tag{4}$$

Here ρ and ν are the density and the kinematic viscosity of the fluid.

The concentration $\theta(x, y, z, t)$ of the contaminant satisfies

$$\frac{\partial\theta}{\partial t} + w\frac{\partial\theta}{\partial z} = \kappa\nabla^2\theta \quad \text{in } S, \tag{5}$$

with

$$\frac{\partial\theta}{\partial n} = 0 \quad \text{on } B, \tag{6}$$

since the pipe is assumed to be impermeable. Here $\partial/\partial n$ denotes the rate of change in the outward normal direction. Equations (5) and (6) have a solution of the form

$$\theta(x, y, z, t) = -\gamma z + \mathrm{Re}\,\{\gamma g(x, y)\,\mathrm{e}^{\mathrm{i}\omega t}\}, \tag{7}$$

provided that

$$\mathrm{i}\omega g - f = \kappa\nabla^2 g \quad \text{in } S, \tag{8}$$

and

$$\frac{\partial g}{\partial n} = 0 \quad \text{on } B. \tag{9}$$

The rate of flux of the contaminant across any plane z = constant is

$$\iint_S \left\{w\theta - \kappa\frac{\partial\theta}{\partial z}\right\} \mathrm{d}x\,\mathrm{d}y = \iint_S [\tfrac{1}{2}(f\,\mathrm{e}^{\mathrm{i}\omega t} + \bar{f}\,\mathrm{e}^{-\mathrm{i}\omega t})\{-\gamma z + \tfrac{1}{2}\gamma(g\,\mathrm{e}^{\mathrm{i}\omega t} + \bar{g}\,\mathrm{e}^{-\mathrm{i}\omega t})\} + \kappa\gamma]\,\mathrm{d}x\,\mathrm{d}y, \tag{10}$$

where the bars denote complex conjugates. The mean rate of flux is therefore

$$\iint_S \{\kappa\gamma + \tfrac{1}{4}\gamma(f\bar{g} + \bar{f}g)\}\,\mathrm{d}x\,\mathrm{d}y. \tag{11}$$

In the absence of any flow this would be $\kappa\gamma A$, where A is the area of the cross-section S. The effective diffusivity in the oscillating flow is thus

$$K = \kappa(1 + R), \tag{12}$$

where

$$R = \frac{1}{4\kappa A}\iint_S (f\bar{g} + \bar{f}g)\,\mathrm{d}x\,\mathrm{d}y. \tag{13}$$

The oscillatory components of frequency 2ω in the flux were studied by Chatwin (1975).

The relative increase R of the flux can also be expressed in terms of a line integral round the boundary B of the cross-section S. The differential equations (3) and (8) show that

$$\bar{g}\nabla^2 f - \left(f + \frac{\kappa P}{\mathrm{i}\omega\rho\nu}\right)\nabla^2\bar{g} - \frac{P}{\omega^2\rho}\nabla^2\bar{f} = \left(\frac{\mathrm{i}\omega}{\nu} + \frac{\mathrm{i}\omega}{\kappa}\right)f\bar{g} + \frac{1}{\kappa}|f|^2 + \frac{P^2}{\omega^2\rho^2\nu}. \tag{14}$$

By applying Green's theorem, combined with the boundary condition (9), to the imaginary part of (14), we find that

$$R = \frac{1}{4i(1 + \sigma^{-1})\,\omega A}\int_B \left\{(\bar{g} + \lambda)\frac{\partial f}{\partial n} - (g + \lambda)\frac{\partial\bar{f}}{\partial n}\right\}\mathrm{d}s, \tag{15}$$

where

$$\sigma = \nu/\kappa \tag{16}$$

is the Schmidt number,

$$\lambda = P/\omega^2\rho, \tag{17}$$

and s is the arclength round the boundary B.

Another useful form for R can be obtained from (13) by elimination of f and $\bar{f}$ through (8). Thus

$$R = \frac{1}{4A}\iint_S (-\bar{g}\nabla^2 g - g\nabla^2\bar{g})\,\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{1}{2A}\iint_S (\operatorname{grad} g)\cdot(\operatorname{grad}\bar{g})\,\mathrm{d}x\,\mathrm{d}y \tag{18}$$

from the divergence theorem and the boundary condition (9). Hence

$$R = \frac{1}{2A}\iint_S \left\{\left|\frac{\partial g}{\partial x}\right|^2 + \left|\frac{\partial g}{\partial y}\right|^2\right\}\mathrm{d}x\,\mathrm{d}y,$$

which shows that R is always positive.

It is convenient to express the functions $f(x,y)$ and $g(x,y)$ in dimensionless forms by the equations

$$f(x,y) = \frac{\mathrm{i}P}{\omega\rho}\{F(x,y)-1\}, \tag{19}$$

$$g(x,y) = \lambda\{G(x,y)-1\}. \tag{20}$$

Then

$$\nabla^2 F = \frac{\mathrm{i}\omega}{\nu}F \quad \text{in } S, \quad F = 1 \quad \text{on } B; \tag{21}$$

$$\nabla^2 G = \frac{\mathrm{i}\omega}{\kappa}(G-F) \quad \text{in } S, \quad \frac{\partial G}{\partial n} = 0 \quad \text{on } B. \tag{22}$$

In terms of F and G, the formulae (15) and (18) become

$$R = \frac{P^2}{4(1+\sigma^{-1})\,\omega^4\rho^2 A}\int_B \left\{G\frac{\partial \bar{F}}{\partial n} + \bar{G}\frac{\partial F}{\partial n}\right\}\mathrm{d}s, \tag{23}$$

$$= \frac{P^2}{2\omega^4\rho^2 A}\iint_S (\operatorname{grad} G)\cdot(\operatorname{grad}\bar{G})\,\mathrm{d}x\,\mathrm{d}y, \tag{24}$$

where the integrals are non-dimensional.

The amplitude of oscillation of the flow has been described so far by that of the pressure gradient, but it may also be expressed in terms of the tidal volume; the rate of working of the pressure gradient, which is equal to the rate of dissipation per unit length of the pipe, is also of interest. The instantaneous flux across any cross-section of the pipe is

$$\iint_S w\,\mathrm{d}x\,\mathrm{d}y = \operatorname{Re}\left\{\frac{\mathrm{i}P}{\omega\rho}\iint_S (F-1)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{e}^{\mathrm{i}\omega t}\right\}.$$

The tidal volume may therefore be defined as

$$V = \frac{2P}{\omega^2\rho}\left|\iint_S (F-1)\,\mathrm{d}x\,\mathrm{d}y\right| \tag{25}$$

$$= \frac{2P}{\omega^2\rho}\left|A + \frac{\mathrm{i}\nu}{\omega}\int_B \frac{\partial F}{\partial n}\mathrm{d}s\right|. \tag{26}$$

The rate at which work is done by the pressure per unit length of the pipe is

$$\iint_S wP\cos\omega t\,\mathrm{d}x\,\mathrm{d}y,$$

so that the mean rate of working is

$$W = \frac{iP^2}{4\omega\rho}\iint_S (F-\bar{F})\,dx\,dy \tag{27}$$

$$= \frac{\nu P^2}{4\omega^2\rho}\int_B \left(\frac{\partial F}{\partial n}+\frac{\partial \bar{F}}{\partial n}\right) ds. \tag{28}$$

The formulae (24) and (26) show that typically

$$R \propto \frac{P^2}{\omega^4\rho^2 d^2}, \quad V \propto \frac{PA}{\omega^2\rho}, \tag{29}$$

where d is a characteristic distance across the pipe. Thus, for a given shape of the cross-section S of the pipe, we can write

$$R = f_S\left(\frac{d^2\omega}{\nu}, \sigma\right)\left(\frac{V}{Ad}\right)^2, \tag{30}$$

where the function f_S depends on the shape of S as well as the dimensionless frequency $d^2\omega/\nu$ and the Schmidt number σ. In §3 we shall study the behaviour of this function for small and large values of the frequency parameter, and for large values of the Schmidt number. Section 4 will give explicit results for the cases of a two-dimensional channel and a circular pipe.

3. Asymptotic forms

When $\omega \to 0$ we can look for regular perturbation expansions

$$F(x,y) = \sum_{r=0}^{\infty}\left(\frac{i\omega}{\nu}\right)^r F_r(x,y), \tag{31}$$

$$G(x,y) = \sum_{r=0}^{\infty}\left(\frac{i\omega}{\nu}\right)^r G_r(x,y). \tag{32}$$

Then

$$\nabla^2 F_0 = \nabla^2 G_0 = 0 \quad \text{in } S, \quad F_0 = 1, \quad \frac{\partial G_0}{\partial n} = 0 \quad \text{on } B; \tag{33}$$

and for $r \geqslant 1$

$$\left.\begin{aligned} &\nabla^2 F_r = F_{r-1}, \quad \nabla^2 G_r = \sigma(G_{r-1}-F_{r-1}) \quad \text{in } S,\\ &F_r = \frac{\partial G_r}{\partial n} = 0 \quad \text{on } B. \end{aligned}\right\} \tag{34}$$

The equations given for G_r do not determine it uniquely, but we must have, for all $r \geqslant 0$,

$$\iint_S (G_r - F_r)\,dx\,dy = 0 \tag{35}$$

in order to be able to solve for G_{r+1}.

From (33) and (35) we find that

$$F_0 = G_0 = 1. \tag{36}$$

Hence

$$\nabla^2 F_1 = 1 \quad \text{in } S, \quad F_1 = 0 \quad \text{on } B. \tag{37}$$

This problem is equivalent to that for steady flow through the pipe. When F_1 has been determined we have

$$G_1 = -L_1/A, \tag{38}$$

where

$$L_1 = -\iint_S F_1\,dx\,dy. \tag{39}$$

If $r \geqslant 2$, G_r is a polynomial in σ of the form

$$G_r(x,y) = \sum_{s=0}^{r-1} G_{rs}(x,y)\,\sigma^s, \tag{40}$$

where G_{r0} is a constant. In particular, G_{21} satisfies

$$\nabla^2 G_{21} = -\left(F_1 + \frac{L_1}{A}\right) \quad \text{in } S, \quad \frac{\partial G_{21}}{\partial n} = 0 \quad \text{on } B. \tag{41}$$

Equation (24) now gives

$$R = \frac{P^2}{2\omega^4\rho^2 A}\iint_S \left\{\sum_{r=2}^{\infty}\left(\frac{\mathrm{i}\omega}{\nu}\right)^r \operatorname{grad} G_r\right\}\cdot\left\{\sum_{r=2}^{\infty}\left(-\frac{\mathrm{i}\omega}{\nu}\right)^r \operatorname{grad} G_r\right\} \mathrm{d}x\,\mathrm{d}y$$

$$= \frac{P^2}{2\rho^2\nu^4 A}\left\{\iint_S (\operatorname{grad} G_2)^2\,\mathrm{d}x\,\mathrm{d}y + O(\omega^2)\right\}.$$

This can be transformed by the divergence theorem to give

$$R = \frac{\sigma^2 P^2}{2\rho^2\nu^4 A}\left\{\iint_S G_{21}\left(F_1 + \frac{L_1}{A}\right)\mathrm{d}x\,\mathrm{d}y + O(\omega^2)\right\}. \tag{42}$$

Although G_{21} is not completely defined by (41), the unknown additive constant is irrelevant.

When $\omega \to 0$, (25) gives

$$V = \frac{2P}{\omega\rho\nu}\{L_1 + O(\omega^2)\}, \tag{43}$$

so that

$$R \sim \frac{\sigma^2}{8AL_1^2}\iint_S G_{21}\left(F_1 + \frac{L_1}{A}\right)\mathrm{d}x\,\mathrm{d}y\left(\frac{\omega V}{\nu}\right)^2. \tag{44}$$

The functions F_r and G_{rs} are proportional to d^{2r}, where d is the typical distance across the pipe. Hence (44) shows that

$$f_S\left(\frac{d^2\omega}{\nu}, \sigma\right) \sim C_S\,\sigma^2\left(\frac{d^2\omega}{\nu}\right)^2 \tag{45}$$

as $\omega \to 0$, where C_S is a number depending only on the shape of the cross-section S.

In the case $\omega \to \infty$, F and G are exponentially small except in a boundary layer on the wall of the pipe. If we assume that the boundary B of the cross-section S has continuous slope and curvature $k(s)$, reckoned positive if the cross-section is convex, we can write

$$\nabla^2 = \frac{1}{1+kn}\left\{\frac{\partial}{\partial s}\left(\frac{1}{1+kn}\frac{\partial}{\partial s}\right) + \frac{\partial}{\partial n}\left((1+kn)\frac{\partial}{\partial n}\right)\right\}, \tag{46}$$

where $n < 0$ in the interior of the pipe. The appropriate boundary-layer variable is

$$\xi = -n\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}} = -n\delta^{-1}, \tag{47}$$

where δ is a measure of the boundary-layer thickness, and then

$$\frac{\partial^2 F}{\partial \xi^2} - \mathrm{i}F = \frac{\delta k}{1-\delta k\xi}\frac{\partial F}{\partial \xi} + \frac{\delta^2}{(1-\delta k\xi)^2}\frac{\partial^2 F}{\partial s^2} - \frac{\delta^3 k'\xi}{(1-\delta k\xi)^3}\frac{\partial F}{\partial s}. \tag{48}$$

The asymptotic solution of (48) as $\omega \to \infty$ is

$$F = F^{(0)} + \delta F^{(1)} + \ldots, \tag{49}$$

where
$$F^{(0)} = e^{-i^{\frac{1}{2}}\xi}, \quad F^{(1)} = \tfrac{1}{2}k\xi\, e^{-i^{\frac{1}{2}}\xi}, \tag{50}$$

and $i^{\frac{1}{2}}$ means $\sqrt{\tfrac{1}{2}}(1+i)$. Similarly we find that
$$G = G^{(0)} + \delta G^{(1)} + \ldots, \tag{51}$$

where
$$G^{(0)} = \frac{\sigma}{\sigma-1}\, e^{-i^{\frac{1}{2}}\xi} - \frac{\delta^{\frac{1}{2}}}{\sigma-1}\, e^{-(i\sigma)^{\frac{1}{2}}\xi}, \tag{52}$$

$$G^{(1)} = \tfrac{1}{2}k\left\{\frac{\sigma}{\sigma-1}\,\xi\, e^{-i^{\frac{1}{2}}\xi} - \left(\frac{i^{\frac{3}{2}}}{\sigma^{\frac{1}{2}}+1} + \frac{\sigma^{\frac{1}{2}}}{\sigma-1}\,\xi\right) e^{-(i\sigma)^{\frac{1}{2}}\xi}\right\} \tag{53}$$

provided that $\sigma \neq 1$, but if $\sigma = 1$ then
$$G^{(0)} = \tfrac{1}{2}(i^{\frac{1}{2}}\xi + 1)\, e^{-i^{\frac{1}{2}}\xi}, \tag{52'}$$

$$G^{(1)} = \tfrac{1}{4}k(i^{\frac{1}{2}}\xi^2 + \xi - i^{\frac{3}{2}})\, e^{-i^{\frac{1}{2}}\xi}, \tag{53'}$$

Using these results in (23), we find that
$$R = \frac{P^2}{4(1+\sigma^{-1})(1+\sigma^{-\frac{1}{2}})\,\omega^4\rho^2 A}\left\{l\left(\frac{2\omega}{\nu}\right)^{\frac{1}{2}} - 2\pi + O(\omega^{-\frac{1}{2}})\right\}, \tag{54}$$

where l is the length of the perimeter of the cross-section. It also follows from (26) that the tidal volume is
$$V = \frac{2P}{\omega^2\rho}\left\{A - l\left(\frac{\nu}{2\omega}\right)^{\frac{1}{2}} + O(\omega^{-1})\right\}, \tag{55}$$

so that
$$R = \frac{l\left(\dfrac{\omega}{2\nu}\right)^{\frac{1}{2}} - \pi + \dfrac{l^2}{A} + O(\omega^{-\frac{1}{2}})}{8(1+\sigma^{-1})(1+\sigma^{-\frac{1}{2}})}\,\frac{V^2}{A^3}. \tag{56}$$

Equation (56) implies that at high frequency we have
$$f_S \sim \frac{1}{8\sqrt{2}}\,\frac{ld}{A}\,\frac{d(\omega/\nu)^{\frac{1}{2}}}{(1+\sigma^{-1})(1+\sigma^{-\frac{1}{2}})}. \tag{57}$$

The results given for $\omega \to \infty$ are not valid if the boundary B of the cross-section has corners, since there will then be boundary regions at the corners, and these will need separate treatment.

The Schmidt number σ is large for diffusion in a liquid. It is therefore appropriate to consider the case in which $\omega d^2/\kappa$ is large but $\omega d^2/\nu$ is not. The equation for $F(x,y)$ must be solved first, and then
$$G(x,y) = \frac{\sigma}{\sigma-1}F(x,y) \tag{58}$$

in the interior of the pipe. The concentration distribution has a boundary layer, which is described by the variable
$$\eta = -n\left(\frac{\omega}{\kappa}\right)^{\frac{1}{2}}, \tag{59}$$

and in this boundary layer
$$G(x,y) = \Gamma_0(s,\eta) + \left(\frac{\kappa}{\omega}\right)^{\frac{1}{2}}\Gamma_1(s,\eta) + \ldots. \tag{60}$$

In the boundary layer we can expand $F(x,y)$ as
$$F(x,y) = 1 - \left(\frac{\partial F}{\partial n}\right)_B \eta\left(\frac{\kappa}{\omega}\right)^{\frac{1}{2}} + O\left(\frac{\kappa}{\omega}\right), \tag{61}$$

and then we find that

$$\Gamma_0 = 1, \quad \Gamma_1 = -\left(\frac{\partial F}{\partial n}\right)_B \{\eta - \mathrm{i}^{\frac{3}{2}} \mathrm{e}^{-\mathrm{i}^{\frac{1}{2}}\eta}\}. \tag{62}$$

Hence from (23)

$$R = \frac{P^2}{4\omega^4\rho^2 A}\int_B \left\{\frac{\partial F}{\partial n} + \frac{\partial \bar{F}}{\partial n} - \frac{\partial F}{\partial n}\frac{\partial \bar{F}}{\partial n}\left(\frac{2\kappa}{\omega}\right)^{\frac{1}{2}} + \ldots\right\} \mathrm{d}s. \tag{63}$$

If further $d^2\omega/\nu \ll 1$, we can use the expansion (29) to obtain

$$R = \frac{P^2}{2\omega^2\rho^2\nu^2 A}\left\{L_1 + \int_B \left(\frac{\partial F_1}{\partial n}\right)^2 \mathrm{d}s \left(\frac{\kappa}{2\omega}\right)^{\frac{1}{2}} + \ldots\right\}, \tag{64}$$

from which

$$R \sim \frac{V^2}{8L_1 A}. \tag{65}$$

4. Special cases

The functions F and G can be found explicitly in two cases: when the pipe is a two-dimensional channel, and when the cross-section is circular.

For the channel $-h < y < h$ we have

$$F(y) = \frac{\cosh y^*}{\cosh h^*}, \tag{66}$$

where

$$y^* = y\left(\frac{\mathrm{i}\omega}{\nu}\right)^{\frac{1}{2}}, \quad h^* = h\left(\frac{\mathrm{i}\omega}{\nu}\right)^{\frac{1}{2}}. \tag{67}$$

Hence when $\sigma \neq 1$

$$G(y) = \frac{\sigma}{\sigma - 1}\left\{F(y) - \frac{\tanh h^*}{\sigma^{\frac{1}{2}}}\frac{\cosh(y^*\sigma^{\frac{1}{2}})}{\sinh(h^*\sigma^{\frac{1}{2}})}\right\}, \tag{68}$$

and if $\sigma = 1$

$$G(y) = \tfrac{1}{2}\operatorname{sech} h^*\{(h^*\coth h^* + 1)\cosh y^* - y^*\sinh y^*\}. \tag{68'}$$

The relative increase of flux can then be calculated from (23). With

$$\beta = h\left(\frac{2\omega}{\nu}\right)^{\frac{1}{2}}, \tag{69}$$

the result can be simplified to

$$R = \frac{4}{\beta^6}\frac{\cosh\beta - \cos\beta}{\cosh\beta + \cos\beta}\frac{C(\beta) - C(\beta\sigma^{\frac{1}{2}})}{1 - \sigma^{-2}}\frac{P^2h^6}{\rho^2\nu^4} \tag{70}$$

provided that $\sigma \neq 1$, where

$$C(\beta) = \frac{\sinh\beta - \sin\beta}{\beta(\cosh\beta - \cos\beta)}. \tag{71}$$

The tidal volume per unit breadth of the channel is

$$V_1 = \frac{4Ph}{\rho\omega^2}\left|1 - \frac{\tanh h^*}{h^*}\right|,$$

so that

$$V_1^2 = \frac{256}{\beta^8}\left\{1 - \frac{2}{\beta}\frac{\sinh\beta + \sin\beta}{\cosh\beta + \cos\beta} + \frac{2}{\beta^2}\frac{\cosh\beta - \cos\beta}{\cosh\beta + \cos\beta}\right\}\frac{P^2h^{10}}{\rho^2\nu^4}. \tag{72}$$

In terms of the tidal volume we therefore have (for $\sigma \neq 1$)

$$R = \frac{\frac{1}{4}\beta^4(\cosh\beta - \cos\beta)}{\beta^2(\cosh\beta + \cos\beta) - 2\beta(\sinh\beta + \sin\beta) + 2(\cosh\beta - \cos\beta)}\frac{C(\beta) - C(\beta\sigma^{\frac{1}{2}})}{1 - \sigma^{-2}}\frac{V_1^2}{(2h)^4}. \tag{73}$$

When $\beta \to 0$

$$C(\beta) = \tfrac{1}{3} - \tfrac{1}{1890}\beta^4 + \tfrac{1}{748440}\beta^8 + O(\beta^{12}), \tag{74}$$

so that

$$R = \frac{\sigma^2}{945}\left\{1 - \frac{5\sigma^2+82}{1980}\beta^4 + O(\beta^8)\right\}\frac{P^2h^6}{\rho^2\nu^4}. \tag{75}$$

Also

$$V_1^2 = \tfrac{64}{9}\{\beta^{-4} - \tfrac{43}{1050} + O(\beta^4)\}\frac{P^2h^{10}}{\rho^2\nu^4}, \tag{76}$$

and therefore

$$R = \frac{\sigma^2}{420}\left\{\beta^{-4} - \frac{175\sigma^2+32}{69300} + O(\beta^4)\right\}\frac{V_1^2}{(2h)^4}, \tag{77}$$

which gives the limiting form

$$R \sim \frac{1}{1680}\left(\frac{V_1\omega}{\kappa}\right)^2. \tag{78}$$

For the high-frequency case we have

$$C(\beta) = \beta^{-1}\{1 + O(e^{-\beta})\} \tag{79}$$

as $\beta \to \infty$. This gives

$$R \sim \frac{4\beta^{-7}}{(1+\sigma^{-1})(1+\sigma^{-\frac{1}{2}})}\frac{P^2h^6}{\rho^2\nu^4} \tag{80}$$

$$\sim \frac{\beta^3}{4(\beta^2-2\beta+2)(1+\sigma^{-1})(1+\sigma^{-\frac{1}{2}})}\frac{V_1^2}{(2h)^4}, \tag{81}$$

where in each case the error is exponentially small as $\beta \to \infty$. If $\beta \to 0$ but $\beta\sigma^{\frac{1}{2}} \to \infty$ we have

$$R = \frac{2}{3\beta^4}\left\{1 - \frac{3}{\beta\sigma^{\frac{1}{2}}} + O(\beta^4, \sigma^{-2})\right\}\frac{P^2h^6}{\rho^2\nu^4} \tag{82}$$

$$\sim \frac{3}{2}\frac{V_1^2}{(2h)^4}. \tag{83}$$

The case $\sigma = 1$ can be treated either directly from (68′) or as a limit from (70). The result is

$$R = \frac{1}{\beta^6}\left\{\frac{\sinh\beta - \sin\beta}{\beta(\cosh\beta + \cos\beta)} + 2\frac{\cosh\beta\cos\beta - 1}{\cosh^2\beta - \cos^2\beta}\right\}\frac{P^2h^6}{\rho^2\nu^4}, \tag{70′}$$

and the corresponding expression in terms of V_1 can be derived from this.

In the case of the circular pipe we take the boundary B as

$$(x^2+y^2)^{\frac{1}{2}} \equiv r = a. \tag{84}$$

Then

$$F(x,y) \equiv F(r) = \frac{I_0(r^*)}{I_0(a^*)}, \tag{85}$$

where

$$r^* = r\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}, \quad a^* = a\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}, \tag{86}$$

and I_0 is the Bessel function. When $\sigma \neq 1$

$$G(x,y) \equiv G(r) = \frac{\sigma}{\sigma-1}\left\{F(r) - \frac{I_0'(a^*)}{\sigma^{\frac{1}{2}}I_0(a^*)}\frac{I_0(r^*\sigma^{\frac{1}{2}})}{I_0'(a^*\sigma^{\frac{1}{2}})}\right\}, \tag{87}$$

and when $\sigma = 1$ this becomes

$$G(r) = \frac{1}{2}\left\{a^*\frac{I_0(r^*)}{I_0'(a^*)} - r^*\frac{I_0'(r^*)}{I_0(a^*)}\right\}. \tag{87′}$$

$\alpha = a\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}}$	$\sigma = \nu/\kappa$				
	0.1	1	10	100	1000
0.1	2.64, −10	2.64, −8	2.64, −6	2.63, −4	1.82, −2
0.2	4.22, −9	4.22, −7	4.22, −5	3.94, −3	5.65, −2
0.5	1.65, −7	1.65, −5	1.60, −3	4.50, −2	8.32, −2
1	2.64, −6	2.62, −4	1.82, −2	7.28, −2	9.23, −2
2	4.15, −5	3.87, −3	5.60, −2	8.76, −2	9.79, −2
5	1.04, −3	3.27, −2	9.19, −2	0.120	0.130
10	4.26, −3	5.46, −2	0.152	0.198	0.213
20	8.55, −3	9.93, −2	0.275	0.358	0.385
50	2.04, −2	0.233	0.645	0.840	0.905
100	3.99, −2	0.457	1.26	1.65	1.77

[p, q denotes $p \times 10^q$.]

TABLE 1. Values of Ra^6/V^2

The increase in mass flux, again calculated from (23), can be expressed most conveniently in terms of the variable

$$\alpha = a\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}} \tag{88}$$

and the Kelvin functions defined by

$$I_0(\alpha i^{\frac{1}{2}}) = \text{ber}(\alpha) + \text{i}\,\text{bei}(\alpha). \tag{89}$$

Let

$$B(\alpha) = \text{ber}^2(\alpha) + \text{bei}^2(\alpha), \tag{90}$$

$$B_1(\alpha) = \frac{\alpha^3 B(\alpha)}{B'(\alpha)}, \tag{91}$$

$$B_2(\alpha) = \frac{\alpha B''(\alpha) + B'(\alpha)}{B'(\alpha)}. \tag{92}$$

Then if $\sigma \neq 1$

$$R = \frac{1 - B_2(\alpha)/B_2(\alpha\sigma^{\frac{1}{2}})}{2(1-\sigma^{-2})\,\alpha^4 B_1(\alpha)} \frac{P^2 a^6}{\rho^2 \nu^4}. \tag{93}$$

The tidal volume is given by

$$V^2 = \frac{4\pi^2 B_3(\alpha)}{\alpha^4 B_1(\alpha)} \frac{P^2 a^{12}}{\rho^2 \nu^4}, \tag{94}$$

where

$$B_3(\alpha) = \frac{\alpha^3 B(\alpha) + B'(\alpha) - \alpha B''(\alpha) - \alpha^2 B'''(\alpha)}{\alpha^4 B'(\alpha)}. \tag{95}$$

Thus

$$R = \frac{1 - B_2(\alpha)/B_2(\alpha\sigma^{\frac{1}{2}})}{8\pi^2(1-\sigma^{-2})\,B_3(\alpha)} \frac{V^2}{a^6}. \tag{96}$$

The corresponding expressions for the case $\sigma = 1$ can be obtained either directly from (87′) or by use of the fact that

$$\lim_{\sigma \to 1}\left(\frac{1 - B_2(\alpha)/B_2(\alpha\sigma^{\frac{1}{2}})}{1-\sigma^{-2}}\right) = \frac{B_1(\alpha) + 2B_2(\alpha) - B_2^2(\alpha) - \alpha^4 B_3(\alpha)}{4B_2(\alpha)}. \tag{97}$$

Values of Ra^6/V^2 are given in table 1 for various values of α and σ. These were computed from the series

$$B(\alpha) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2}\alpha)^{4n}}{(n!)^2\,(2n)!} \tag{98}$$

and the asymptotic form as $\alpha \to \infty$

$$B(\alpha) \sim \frac{1}{2\pi\alpha} \exp\left\{\sqrt{2}\,\alpha + \frac{\sqrt{2}}{8\alpha} - \frac{25\sqrt{2}}{384\alpha^3} - \frac{13}{64\alpha^4} + \dots\right\} \quad (99)$$

(see Watson 1944, §§5.41, 7.24). These expressions can also be used to study the behaviour of R for small and large values of α.

When $\alpha \to 0$ we find from (98) that

$$R = \frac{\sigma^4}{6144}\left\{1 - \frac{13\sigma^2+88}{2880}\alpha^4 + O(\alpha^8)\right\}\frac{P^2a^6}{\rho^2\nu^4} \quad (100)$$

$$= \frac{\sigma^2\alpha^4}{384\pi^2}\left\{1 - \frac{13\sigma^2+3}{2880}\alpha^4 + O(\alpha^8)\right\}\frac{V^2}{a^6}. \quad (101)$$

When $\alpha \to \infty$, the asymptotic formula (99) gives

$$R = \frac{\sqrt{2} - \alpha^{-1} + O(\alpha^{-2})}{2\alpha^7(1+\sigma^{-1})(1+\sigma^{-\frac{1}{2}})}\frac{P^2a^6}{\rho^2\nu^4} \quad (102)$$

$$= \frac{\alpha + \frac{3}{2}\sqrt{2} + O(\alpha^{-1})}{4\sqrt{2}\,\pi^2(1+\sigma^{-1})(1+\sigma^{-\frac{1}{2}})}\frac{V^2}{a^6}. \quad (103)$$

The case treated by Farrell & Larson (1973) is that of a circular pipe in which the oscillations are slow, so that the velocity profile approximates to that of steady flow, but the diffusivity κ is small. The parameter characterizing this problem is $a(\omega/\kappa)^{\frac{1}{2}} = \alpha\sigma^{\frac{1}{2}}$. When $\alpha \to 0$ in (96) with $\alpha\sigma^{\frac{1}{2}}$ remaining fixed

$$R = \left\{1 - \frac{4}{B_2(\alpha\sigma^{\frac{1}{2}})} + \frac{\alpha^4}{1152}\left(1 - \frac{16}{B_2(\alpha\sigma^{\frac{1}{2}})}\right) + O(\alpha^8)\right\}\frac{V^2}{\pi^2(1-\sigma^{-2})a^6}, \quad (104)$$

and the leading term of this expression agrees with the result of Farrell & Larson. In the limit when $\alpha\sigma^{\frac{1}{2}} \to \infty$, but $\alpha \to 0$, we have

$$R = \frac{1}{\pi^2}\left\{1 - \frac{2\sqrt{2}}{\alpha\sigma^{\frac{1}{2}}} + \frac{3\sqrt{2}}{4(\alpha\sigma^{\frac{1}{2}})^3} + O\left(\alpha^4, \frac{1}{\alpha^4\sigma^2}\right)\right\}\frac{V^2}{a^6}. \quad (105)$$

It may be verified that the asymptotic forms, both for the channel and for the circular pipe, agree with the general results of §3.

5. Conclusion

When there is, on average, a linear gradient of the concentration of a passive contaminant in an oscillating flow in a pipe, the effective diffusivity of the contaminant is

$$K = \kappa(1+R), \quad (12)$$

where the relative increase R is of the form

$$R = f_S\left(\frac{d^2\omega}{\nu}, \sigma\right)\left(\frac{V}{Ad}\right)^2. \quad (30)$$

Here the dimensionless forms of the frequency ω and the tidal volume V involve d, a typical distance across the pipe. As in Taylor's problem of steady flow, when a small quantity of the diffusing substance is introduced into the pipe, the later stages of the dispersion of the substance will be governed by the diffusivity K. The dispersion of the contaminant at earlier times presents a more difficult problem, which has recently been studied by Smith (1982) as a delay-diffusion process.

The function f_S, which depends on the shape of the cross-section S as well as the dimensionless frequency and Schmidt number, increases with frequency, ultimately as its square root. The flux of the contaminant can therefore be increased significantly, for given tidal volume, by the use of a high frequency of oscillation. The price for this is that the work done by the pressure gradient also increases with frequency. It follows from (27) and (28) that at low frequency the mean rate of working

$$W \sim \frac{L_1 P^2}{2\rho\nu} \sim \frac{\rho\nu\omega^2 V^2}{8L_1}, \tag{106}$$

whereas for high frequencies

$$W \sim \tfrac{1}{2}l\left(\frac{\nu}{2\omega}\right)^{\frac{1}{2}}\frac{P^2}{\rho\omega} \sim \tfrac{1}{8}l\left(\frac{\omega}{2\nu}\right)^{\frac{1}{2}}\frac{\rho\nu\omega^2 V^2}{A^2}, \tag{107}$$

where L_1 is defined in (39) and l is the length of the perimeter of the cross-section. The energy thus produced is dissipated throughout the pipe at low frequency, but only in the boundary layer at high frequency. This corresponds to the fact that the increase of flux arises from the whole flow at low frequency, but only from the boundary layer at high frequency.

Experiments by Joshi *et al.* (1983), carried out on the diffusion of methane in air in a circular pipe, have confirmed the prediction that R is proportioned to V^2. They also give reasonable agreement with the computed variation of the function f_S with frequency. Numerical and experimental studies have been made by Stairmand (1983) for the case of a flat channel, and these also confirm that R is proportional to V^2.

Dispersion in oscillatory flow is important in many practical cases, ranging from tidal flow in estuaries to respiratory flow in the airways of the lungs. In many cases a steady component is also present in the basic flow. When the pressure gradient is

$$\frac{\partial p}{\partial x} = -P\cos\omega t - P_0, \tag{108}$$

the velocity distribution at large times is

$$w = \mathrm{Re}\{f\mathrm{e}^{\mathrm{i}\omega t}\} - \frac{P_0}{\rho\nu}F_1(x,y), \tag{109}$$

where $F_1(x,y)$ is defined by (37). The mean volume flux is therefore

$$Q = \frac{P_0 L_1}{\rho\nu}. \tag{110}$$

Equation (7) must be replaced by

$$\theta = -\gamma\bar{z} + \mathrm{Re}\{\gamma g\,\mathrm{e}^{\mathrm{i}\omega t}\} + h(x,y), \tag{111}$$

where

$$\bar{z} = z - \frac{Qt}{A}, \tag{112}$$

$$h(x,y) = -\frac{\gamma Q}{\kappa L_1}G_{21}(x,y). \tag{113}$$

The flux of the contaminant across the moving plane $\bar{z}$ = constant is

$$\iint_S \left\{\left(w - \frac{Q}{A}\right)\theta - \kappa\frac{\partial\theta}{\partial z}\right\}\mathrm{d}x\,\mathrm{d}y, \tag{114}$$

and the mean value of this is

$$\iint_S \left\{\kappa\gamma + \tfrac{1}{4}\gamma(f\bar{g}+\bar{f}g) + \frac{\gamma Q^2}{\kappa L_1^2} G_{21}\left(F_1 + \frac{L_1}{A}\right)\right\} \mathrm{d}x\,\mathrm{d}y. \tag{115}$$

Hence the effective longitudinal diffusivity is now

$$K = \kappa(1+R+R_0), \tag{116}$$

where

$$R_0 = \frac{Q^2}{\kappa^2 L_1^2 A} \iint_S G_{21}\left(F_1 + \frac{L_1}{A}\right) \mathrm{d}x\,\mathrm{d}y. \tag{117}$$

If there is no oscillatory component of the flow, we recover the result of Taylor's theory. Thus (116) shows that the effects of steady flow and oscillatory flow are additive. It will be observed that (44) shows that in the case of slow oscillations of the flow

$$R \sim \frac{1}{8}\left(\frac{\omega V}{Q}\right)^2 R_0, \tag{118}$$

a result which generalizes that of Bowden (1965) for simple shear flow.

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